

Ground state energy of unitary fermion gas with the Thomson Problem approach

Ji-sheng Chen ^{*}

*Institute of Particle Physics & Physics Department,
Central China Normal University, Wuhan 430079, People's Republic of China*

The dimensionless universal coefficient ξ defines the ratio of the unitary fermions energy density to that for the ideal non-interacting ones in the non-relativistic limit with $T = 0$. The classical Thomson Problem is taken as a nonperturbative quantum many-body arm to address the ground state energy including the low energy nonlinear quantum fluctuation/correlation effects. With the relativistic Dirac continuum field theory formalism, the concise expression for the energy density functional of the strongly interacting limit fermions at both finite temperature and density is obtained. Analytically, the universal factor is calculated to be $\xi = \frac{4}{9}$. The energy gap is $\Delta = \frac{5}{18}k_f^2/(2m)$.

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With the further developments of the Bardeen-Cooper-Schrieffer theory, the possibility about the existence of the fermions superfluidity in the dilute gas system motivates widely theoretical studies and experimental efforts. Since DeMarco and Jin achieved the Fermi degeneracy[1], the ultra-cold fermion atoms gas has stirred intense interest about the fundamental Fermi-Dirac statistical physics in the strongly interacting limit.

Across the Feshbach resonance regime, the interaction changes from weakly to strongly attractive according to the magnitude of the magnetic field. At the midpoint of this crossover unitary regime from Bardeen-Cooper-Schrieffer(BCS) to Bose-Einstein condensation(BEC), the scattering length will diverge due to the existence of a zero-energy bound state for the two-body system. In this limit, the only dimensionful parameter is the Fermi momentum k_f at $T = 0$. The corresponding energy scale is the Fermi kinetic energy $\epsilon_f = k_f^2/(2m)$, while m is the fermion mass. According to dimensional analysis, the system details do not contribute to the thermodynamics properties, i.e., the thermodynamics properties are universal [2–19]. The energy density should be proportional to that of a free Fermi gas $E/V = \xi(E/V)_{free} = \xi\frac{3}{5}n\epsilon_f$.

This fundamental dimensionless universal constant ξ has attracted much attention theoretically/experimentally in recent years. Various theoretical approaches have been tried and the results differ from each other remarkably, for example, see Ref.[2–19] and references therein. The theoretical calculations are about $\xi \sim 0.3 - 0.6$. Intriguingly, the experimental results are also quite different from each other, for example, $\xi \approx 0.74 \pm 0.07$ [20], $\xi \approx 0.7$ [21], $\xi = 0.32^{+0.13}_{-0.10}$ [22], $\xi = 0.51 \pm 0.04$ [23], $\xi = 0.46 \pm 0.05$ [24], $\xi = 0.46^{+0.05}_{-0.12}$ [25]. The recent lattice result is $\xi = 0.25 \pm 0.03$ [26].

How to approach the exact value of ξ analytically is a bewitching topic in the Fermi-Dirac statistical physics. To attack this intriguing topic is a seriously difficult problem in many-body theory. The essential task is how to in-

corporate the nonlinear quantum fluctuation/correlation effects into the thermodynamics by going beyond any naive loop diagram expansions or the lowest order mean field theory. To our knowledge, the hitherto considerations looking for ξ have been solely limited in the non-relativistic frameworks and with quite different results. How about a relativistic Dirac phenomenology attempt?

Motivation: Essentially, the unitary physics with infinite scattering lengths is quite similar to the *universal* strongly instantaneous Coulomb correlation thermodynamics in a compact nuclear confinement environment resulting from the competition of long and short range forces[27]. At the crossover point, the cross-section between the two-body particle is limited by $\sigma \sim 4\pi/k^2$ (k is the relative wavevector of the colliding particles), while the gauge vector boson propagator is $\Delta_{\mu\nu} \sim g_{\mu\nu}/k^2$ in the gauge field theory. The former is short range but exact long range infrared correlation while the latter is long range one. The analogism motivates us to use the latter to model the former.

In this context, it is also instructive to recall the intermediate vector boson (IVB) hypothesis in weak interaction theory. The local intermediate vector boson theory is related with the current-current(CC) contact interaction version through the corresponding connection: $g_W^2/m_W^2 \equiv \frac{G}{\sqrt{2}}$. In the low energy limit $k \rightarrow 0$, the two IVB and CC theories are identical. To model the unitary limit, we “let” $g_W^2/m_W^2 \rightarrow \infty$ with an arbitrary large charge g_W or with an arbitrary small mass gap m_W .

It is usually assumed that there exists a uniform opposite charged background to ensure the stability in discussing the electrons system thermodynamics. This is the well-known classical Thomson Problem[28]. Of course, there is not a finished standard answer for itself/generalized version for over a century[29, 30]. Furthermore, a consummate method to gauge the infrared singularity in the dense and hot gauge theory is still to be looked for. However, we will turn the unitary limit topic into the same involved infrared problem. Then, the Thomson Problem is used as a potential quantum many-body nonperturbative arm to attack the infrared one.

^{*}chenjs@iopp.ccnu.edu.cn

To achieve the intriguing physics with the unusual in-medium relativistic Lorentz invariance breaking at unitarity, an infrared correlating “QED” Lagrangian is proposed in this *Letter*. Let the fermion have an “electric” charge g in addition to other internal global $U(1)$ symmetry quantum numbers. According to the general stability principle, the system should be stabilized by a fictive uniform opposite charged background in the meantime. This particular assumption makes it possible for us to deal with the challenging infinity.

The natural units $c = \hbar = k_B = 1$ are used.

To perform the path integral presentation as a nonperturbative starting point, the considered effective actions involve the interaction of Dirac fermions with an auxiliary Proca-like Lorentz vector boson field[31, 32]

$$\begin{aligned}\mathcal{L}_{\text{matter}} &= \bar{\psi}(i\gamma_\mu\partial^\mu - m)\psi; \\ \mathcal{L}_{A,\text{free}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}; \\ \mathcal{L}_{I,A-\psi} &= \frac{1}{2}m_{\text{background}}^2 A_\mu A^\mu + A_\mu J^\mu,\end{aligned}\quad (1)$$

where m is the bare fermion mass. The A_μ is the vector field with the field stress

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (2)$$

In the action $\mathcal{L}_{I,A-\psi}$, the electric vector current contributed by the fermions is

$$J^\mu = g\bar{\psi}\gamma^\mu\psi. \quad (3)$$

Based on the local gauge invariant free Lagrangian, the many-body interactions can be introduced with a hidden local symmetry (HLS) manner[33, 34]

$$\mathcal{L}_I = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + |D_\mu H|^2 + V(H) + A_\mu J^\mu. \quad (4)$$

The many-body Lorentz violation environment modulates the bare two-body interaction. The vector boson mode mass gap $m_{\text{background}}$ is not an internal degree of freedom. It appears as a free parameter and controls the interaction strength between the particles. It indicates the interaction of the opposite electric charged background or the stochastic many-body potential caused by the strongly fluctuating/correlating effects. Below, the suffix “background” will be emitted for brevity just with a tilde symbol $m_{\tilde{B}}$. The un-physical coupling constant g also represents the unitary limit/infrared characteristic “two-body” bare potential.

The electric current conservation/gauge invariance is guaranteed by the Lorentz transversality condition with HLS formalism

$$\partial_\mu A^\mu = 0, \quad (5)$$

which can be naturally realized by taking the relativistic Hartree instantaneous approximation (RHA).

In terms of the functional path integral[27, 35–37], the auxiliary effective potential reads

$$\Omega/V = -\frac{1}{2}m_{\tilde{B}}^2 A_0^2 - 2T \int_k \left[\ln(1 + e^{-\beta(E_k - \mu^*)}) + \ln(1 + e^{-\beta(E_k + \mu^*)}) \right], \quad (6)$$

where “2” represents the (hyperfine-)spin projection of the fermions with $\int_k = \int d^3\mathbf{k}/(2\pi)^3$ and $\beta = 1/T$ being the inverse temperature. The tadpole diagram with the boson self-energy for the full fermion propagator leads to

$$A_0 = -\frac{g}{m_{\tilde{B}}^2}n, \quad (7)$$

with the fermions (electric charge number) density defined by the thermodynamics relation $\partial\Omega/\partial\mu|_{A_0} \equiv -n$

$$n = 2 \int_k [f(\mu^*, T) - \bar{f}(\mu^*, T)]. \quad (8)$$

In the above expressions,

$$f(\mu^*, T) = \frac{1}{e^{\beta(E_k - \mu^*)} + 1}, \quad \bar{f}(\mu^*, T) = \frac{1}{e^{\beta(E_k + \mu^*)} + 1},$$

are the distribution functions for (anti-)particles with $E_k = \sqrt{\mathbf{k}^2 + m^2}$. From Eq.(7), the effective chemical potential μ^* is defined with a gauge invariant manner

$$\mu^* \equiv \mu + \mu_I = \mu - \frac{g^2}{m_{\tilde{B}}^2}n, \quad (9)$$

where μ is the global chemical potential. The spirit is quite similar to the Kohn-Sham density functional theory[38].

Using Eq.(6) and with the thermodynamics relation

$$\epsilon = \frac{1}{V} \frac{\partial(\beta\Omega)}{\partial\beta} + \mu n, \quad (10)$$

one obtains the energy density functional

$$\epsilon = \frac{1}{2} \frac{g^2}{m_{\tilde{B}}^2} n^2 + 2 \int_k E_k [f(\mu^*, T) + \bar{f}(\mu^*, T)]. \quad (11)$$

The second term in Eq.(11) appears as very much the analytical formalisms for the free Fermi-Dirac gas. However, the correlating effects are implicitly included through the effective chemical potential esp. for $T \neq 0$.

The mass gap parameter $m_{\tilde{B}}^2$ is a Lagrange *multiplier* that enforces relevant constraints and reflects quantum fluctuating effects consequently. The remaining central task is how to determine the unknown many-body stochastic potential characterized by the coupling constants $m_{\tilde{B}}^2$ and g^2 . The answer can be found from the auxiliary effective potential, i.e., the composite system should be “charge neutralized” through the fictive background with the artificial conditional extremum [39]

$$\frac{\partial\Omega}{\partial\mu}|_{m_{\tilde{B}}^2, T} = 0. \quad (12)$$

With Eq.(12), one can have

$$m_{\tilde{B}}^2 = -\frac{g^2}{\pi^2} \int_0^\infty d|\mathbf{k}| \frac{(2\mathbf{k}^2 + m^2)}{E_k} [f(\mu^*, T) + \bar{f}(\mu^*, T)] \equiv -m_D^2, \quad (13)$$

which is the *negative* of the gauge invariant Debye (Thomas-Fermi) screening mass squared m_D^2 , i.e., there is a minus sign between $m_{\tilde{B}}^2$ and m_D^2 . The Debye mass parameter can be also directly calculated with the vector boson polarization through the Dyson-Schwinger equation (relativistic random phase approximation-RPA)[40]

$$\Pi_A^{\mu\nu}(p_0, \mathbf{p}) = g^2 T \sum_{k_0} \int_k \text{Tr} \left[\gamma^\mu \frac{1}{\not{k} - m} \gamma^\nu \frac{1}{(\not{k} - \not{p}) - m} \right], \quad (14)$$

with the full fermion propagator Eq.(9) by noting

$$m_D^2 = -\Pi_A^{00}(0, |\mathbf{p}| \rightarrow 0). \quad (15)$$

In Eq.(14), the 0-component of the four-momentum $k = (k_0, \mathbf{k})$ in the fermion loop is related to temperature T and effective chemical potential μ^* via $k_0 = (2n+1)\pi Ti + \mu^*$. It is very interesting that the Thomson stability condition can give the gauge invariant Debye mass, which is *exactly* consistent with the Dyson-Schwinger equation.

At $T = 0$, the parameter $m_{\tilde{B}}^2$ is

$$m_{\tilde{B}}^2 = -\frac{g^2}{\pi^2} k_f E_f, \quad (16)$$

with k_f being the Fermi momentum and $E_f = \sqrt{k_f^2 + m^2}$ the relativistic Fermi kinetic energy.

The Eq.(11) with Eq.(13) ($m_{\tilde{B}}^2 = -m_D^2$) are our main results, from which one can further study the strongly correlating fermions thermodynamics in the unitary limit. The collective interaction contribution is *negative* for the physical energy density functional. Especially, there is not any remained parameter because the un-physical coupling constant g appears simultaneously in the denominator and numerator within the relevant analytical expressions through $m_{\tilde{B}}^2$. From Eq.(16), one can see that the magnitude of this fraction ratio $m_{\tilde{B}}^2/g^2$ characterize the density of states and consequently the fluctuating contributions.

With the mathematically well defined energy density functional Eq.(11) and with Eq.(13), we now return to the final discussions. At $T \rightarrow 0$, the energy density is

$$\epsilon = -\frac{k_f^5}{18\pi^2 E_f} + \frac{(2k_f^2 + m^2)k_f E_f - m^4 \ln \frac{k_f + E_f}{m}}{8\pi^2}. \quad (17)$$

In the non-relativistic limit $k_f/m \ll 1$, one can expand ϵ according to the Taylor series of k_f/m

$$\epsilon = mn + \frac{4}{15}n\varepsilon_f + \frac{5}{42}n\frac{\varepsilon_f^2}{m} - \frac{1}{3}n\frac{\varepsilon_f^3}{m^2} + \dots, \quad (18)$$

with the Fermi kinetic energy $\varepsilon_f = k_f/(2m)$ and particle number density $n = k_f^3/(3\pi^2)$. Therefore, one can obtain the ground state binding energy (energy per particle)

$$e_b = \frac{\epsilon}{n} - m = \left(\frac{4}{15} + \frac{5}{42} \frac{\varepsilon_f}{m} - \frac{1}{3} \left(\frac{\varepsilon_f}{m} \right)^2 + \dots \right) \varepsilon_f. \quad (19)$$

By keeping up to the lowest order of k_f/m , the ratio of the binding energy to the Fermi kinetic energy is $\frac{4}{15}$. Furthermore, one can obtain the the universal dimensionless coefficient ξ , i.e, the ratio of the energy density to that of a free Fermi gas

$$\xi = \frac{E}{N} / \left(\frac{E}{N} \right)_{free} = \frac{4}{15} \times \frac{5}{3} = \frac{4}{9}. \quad (20)$$

This analytical result is in the range of the existed theoretical ones $\xi \sim 0.3 - 0.6$ [4–19]. It is exactly consistent with that of the quantum Monte Carlo calculations[5] and in reasonable agreement with the updating experiments[22–25]. We also note a similar $\xi_{D=\infty} = \frac{4}{9}$ was obtained by Steele within the effective theory long ago, with D being the space-time dimensions[41].

At unitarity, the energy gap should be also proportional to the Fermi kinetic energy according to the dimensional analysis. It can be *empirically* derived from the total energy density with the usual odd-even staggering in the thermodynamics limit[5, 11] $\Delta = \frac{5}{18}\varepsilon_f$, with ε_f being the Fermi kinetic energy. The critical temperature is $T_c \approx 0.157T_F$ approximated with the BCS universal relation $T_c = e^\gamma \Delta/\pi$, where γ is the Euler constant. The T_c is in agreement with the updating theoretical[19, 42]/experimental[23, 43] results. It should be pointed out that the differences for the energy gap Δ /critical BCS phase transition temperature T_c can be as large as several times in the literature. In the meantime, the validity of the weak coupling BCS relation between Δ and T_c deserves to be further studied theoretically.

Let us further analyze the pressure by keeping up only to the lowest order of k_f

$$P = \frac{1}{30m} (3\pi^2)^{\frac{2}{3}} n^{1+\frac{2}{3}} = \alpha n^{1+\frac{2}{3}}. \quad (21)$$

The power index of n is also consistent with the existed theoretical calculations[44]. One can see that the strongly correlating effect as well as the fermion mass affects the coefficient α significantly.

The sound speed squared for the ideal Fermi gas is

$$v_{FG}^2 = \frac{1}{m} \frac{\partial P_{FG}}{\partial n} = \frac{1}{3} v_f^2, \quad (22)$$

with the Fermi velocity $v_f = k_f/m$. In the unitary limit, the pressure is $P = \frac{1}{6}P_{FG}$, from which one can find the sound speed is reduced remarkably

$$v = \sqrt{\frac{1}{6}v_{FG}} = \frac{\sqrt{2}}{6}v_f > 0. \quad (23)$$

It has been argued that the sound speed squared might be negative due to the theoretical spinodal instability in the unitary limit[4]. Although the sound speed is significantly reduced due to the correlating effects, it is still a real number which indicates the system stability.

The *interior* correlations have been refreshingly incorporated in the thermodynamics as an *external* source manner indirectly through a mirror Thomson background. In other words, we have derived the effective interaction strength, with which we have given the general but concise analytical thermodynamic expressions at both finite temperature and density. This analytic method can be easily extended to the near unitary limit regime[45].

In conclusion, there is a similarity between the strongly interacting ultra-cold atoms physics in the scattering length limit $|a| \rightarrow \infty$ and the infinite instantaneous Coulomb interaction in a compact confinement environment due to the competition of long range forces with short range ones. The classical Thomson Problem as a potential quantum many-body arm redounds to addressing the universal thermodynamics, with which the key coefficient $\xi = \frac{4}{9}$ and the energy gap $\Delta = \frac{5}{18}\varepsilon_f$ are obtained in the QED-like framework with HLS formalism.

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